

## Non-perturbative renormalization of kaon four-quark operators with $n_f = 2 + 1$ Domain Wall fermions

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**Peter Boyle, Nicolas Garron\* for the RBC-UKQCD collaborations.**

*School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, U.K.*

*E-mail: [nicolas.garron@ed.ac.uk](mailto:nicolas.garron@ed.ac.uk)*

We present our strategy and some preliminary results for the renormalization of four-quark operators relevant for kaon physics. We follow the non-perturbative Rome-Southampton method, with both exceptional and non-exceptional kinematics. We also implement momentum sources and twisted boundary conditions. We use an (almost) unitary setup: Domain-Wall valence on  $n_f = 2 + 1$  Domain-Wall sea and Iwasaki gauge action, at two values of the lattice spacing corresponding to approximately 0.086 fm and 0.114 fm. The chiral properties of these fermions play a crucial role in this computation and are studied in detail in this work.

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\*Speaker.

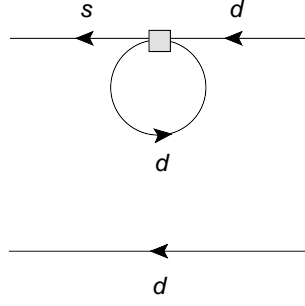
## 1. Introduction

Kaon physics has been extensively studied through lattice simulation for more than thirty years, and thanks to the recent algorithms and hardware developments one can now achieve the precision required to constraint the standard model and hopefully reveal the effect of new physics. Recently,  $B_K$ , the quantity which parametrizes neutral kaon mixing in the standard model has been computed with an accuracy of a few percents [1]. Nevertheless, other kaon matrix elements are still poorly determined although they can have a great impact in the search for new physics or imply strong constraints on beyond the standard model (BSM) theories. For example, the non-perturbative contributions to neutral kaon mixing beyond the standard model have been computed only in the quenched approximation [2, 3], although computation with dynamical fermions are currently underway and some preliminary studies have been already presented [4, 5]. Probably even more importantly, a complete computation of  $K \rightarrow \pi\pi$  decays with dynamical quarks is still missing. Because the experimental parameters of CP violations are very well measured ( $|\epsilon| = (2.228 \pm 0.011) \times 10^{-3}$  and  $\text{Re}(\epsilon'/\epsilon) = (1.65 \pm 0.26) \times 10^{-3}$  [6]), a precise and realistic computation of the relevant matrix elements would provide important constraints on the CKM matrix. One of the difficulties for the lattice implementation comes from the two-body final state. In the past this problem was usually circumvented by invoking the soft pion theorem to relate the two-pion state to a one-pion state. But, as it has been shown in [7], this approach is not reliable for a precise computation, mainly because of the poor convergence of chiral perturbation theory at masses around those of the kaon. A very important step forward has been made when it was realized how the energy shift of a two-particle state can be computed on the lattice [8]. The RBC-UKQCD collaborations have started the computation of this decay along this line, and the first results for the matrix element of the  $\Delta I = 3/2$  operators have been presented at this conference [9]. An alternative method has been recently presented in [10]. In this proceeding we present our strategy and some preliminary results for the renormalization of some of the relevant four-quark operators. It has become traditional to use a non-perturbative renormalization scheme like the Schrödinger functional or the RI-MOM scheme. Here we use a modified version of the latter, following what was done recently for  $B_K$  [1], but generalized to other four-quark operators. In the next section we explain what are the operators we consider in this work. In the third section, we give more details about the numerical techniques, and preliminary results are presented in the fourth section.

## 2. General Framework

### 2.1 Kaon decay

In the standard model at an energy scale below the charm quark mass, the dominant non perturbative contributions to the effective  $\Delta s = 1$ ,  $\Delta d = -1$  Hamiltonian can be described by a linear combination of ten four-quark operators: two current-current, four QCD penguins, and four electroweak penguins. Among these 10 operators, only seven are actually independent and it is useful to classify them according to their chiral and isospin properties. Hence one notices that they fall into three different representations of  $SU(3)_L \times SU(3)_R$  which are  $(27, 1)$ ,  $(8, 1)$  and  $(8, 8)$ , and they can contribute to two isospin channels  $\Delta I = 3/2$  and  $\Delta I = 1/2$  (see for example [11, 12]). We will use the seven-operator renormalization basis defined in [12], in which the operators have



**Figure 1:** Example of eye diagram which contribute to  $K \rightarrow \pi\pi$  decay in the  $\Delta I = 1/2$  channel.

the following properties:

Operator	$SU(3)_L \times SU(3)_R$	$\Delta I$
$Q'_1$	$(27, 1)$	$1/2, 3/2$
$Q'_2, Q'_3, Q'_5, Q'_6$	$(8, 1)$	$1/2$
$Q'_7, Q'_8$	$(8, 8)$	$1/2, 3/2$

(2.1)

If chiral symmetry was exact, the operators of different chirality would not mix under renormalization, and in the basis described above the renormalization matrix would take the block diagonal form:

$$Z^{\Delta S=1} = \begin{pmatrix} Z_{11}^{\Delta S=1} & & & & & & & & & \\ & Z_{22}^{\Delta S=1} & Z_{23}^{\Delta S=1} & Z_{25}^{\Delta S=1} & Z_{26}^{\Delta S=1} & & & & & \\ & Z_{32}^{\Delta S=1} & Z_{33}^{\Delta S=1} & Z_{35}^{\Delta S=1} & Z_{36}^{\Delta S=1} & & & & & \\ & Z_{52}^{\Delta S=1} & Z_{53}^{\Delta S=1} & Z_{55}^{\Delta S=1} & Z_{56}^{\Delta S=1} & & & & & \\ & Z_{62}^{\Delta S=1} & Z_{63}^{\Delta S=1} & Z_{65}^{\Delta S=1} & Z_{66}^{\Delta S=1} & & & & & \\ & & & & & Z_{77}^{\Delta S=1} & Z_{78}^{\Delta S=1} & & & \\ & & & & & Z_{87}^{\Delta S=1} & Z_{88}^{\Delta S=1} & & & \end{pmatrix}. \quad (2.2)$$

Furthermore, since isospin is an exact symmetry in the chiral limit, for the  $(27,1)$  and  $(8,8)$  operators, it is enough to consider only the  $\Delta I = 3/2$  parts. This is numerically advantageous because the “eye diagram” (see fig. 1) which are difficult to compute can only contribute to  $\Delta I = 1/2$  processes.

### 3. Neutral kaon mixing

In the standard model, neutral kaon mixing is dominated by box diagrams like the one shown in figure 2. The non-perturbative contributions are given by  $\langle \bar{K}^0 | O_{VV+AA}^{\Delta S=2} | K^0 \rangle$ , where  $O_{VV+AA}^{\Delta S=2}$  is the parity conserving part of  $(\bar{s}\gamma_\mu^L d)(\bar{s}\gamma_\mu^L d)$ . Beyond the standard model, other operators contribute and they are usually given in the so-called SUSY basis

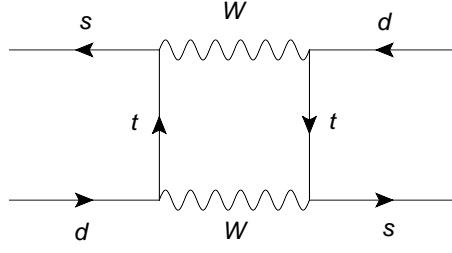
$$O_1^{\Delta S=2} = (\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) d_\alpha) (\bar{s}_\beta \gamma_\mu (1 - \gamma_5) d_\beta), \quad (3.1)$$

$$O_2^{\Delta S=2} = (\bar{s}_\alpha (1 - \gamma_5) d_\alpha) (\bar{s}_\beta (1 - \gamma_5) d_\beta), \quad (3.2)$$

$$O_3^{\Delta S=2} = (\bar{s}_\alpha (1 - \gamma_5) d_\beta) (\bar{s}_\beta (1 - \gamma_5) d_\alpha), \quad (3.3)$$

$$O_4^{\Delta S=2} = (\bar{s}_\alpha (1 - \gamma_5) d_\alpha) (\bar{s}_\beta (1 + \gamma_5) d_\beta), \quad (3.4)$$

$$O_5^{\Delta S=2} = (\bar{s}_\alpha (1 - \gamma_5) d_\beta) (\bar{s}_\beta (1 + \gamma_5) d_\alpha). \quad (3.5)$$



**Figure 2:** Example of box diagram contributing to  $K - \bar{K}$  mixing in the Standard model.

In this basis  $O_1^{\Delta s=2}$  is the standard model operator and  $O_i^{\Delta s=2}, i > 1$  are the BSM ones. In the  $SU(3)$  flavor limit, It is straightforward to relate  $O_4^{\Delta s=2}$  and  $O_5^{\Delta s=2}$  to the  $\Delta I = 3/2$  components of the electroweak penguins  $Q'_7$  and  $Q'_8$ , which transform under  $(8, 8)$ . As one can find out from their flavor structures, the remaining operators  $O_2^{\Delta s=2}$  and  $O_3^{\Delta s=2}$  transform under  $(6, \bar{6})$ <sup>1</sup>. Thus, if chiral symmetry is respected,  $O_1^{\Delta s=2}$  renormalizes multiplicatively,  $O_2^{\Delta s=2}$  and  $O_3^{\Delta s=2}$  mix together, and so do  $O_4^{\Delta s=2}$  and  $O_5^{\Delta s=2}$ . To simplify the numerical implementation we work in the following basis:

$$Q_1^{\Delta s=2} = (\bar{s}_\alpha \gamma_\mu d_\alpha)(\bar{s}_\beta \gamma_\mu d_\beta) + (\bar{s}_\alpha \gamma_\mu \gamma_5 d_\alpha)(\bar{s}_\beta \gamma_\mu \gamma_5 d_\beta), \quad (3.6)$$

$$Q_2^{\Delta s=2} = (\bar{s}_\alpha \gamma_\mu d_\alpha)(\bar{s}_\beta \gamma_\mu d_\beta) - (\bar{s}_\alpha \gamma_\mu \gamma_5 d_\alpha)(\bar{s}_\beta \gamma_\mu \gamma_5 d_\beta), \quad (3.7)$$

$$Q_3^{\Delta s=2} = (\bar{s}_\alpha d_\alpha)(\bar{s}_\beta d_\beta) - (\bar{s}_\alpha \gamma_5 d_\alpha)(\bar{s}_\beta \gamma_5 d_\beta), \quad (3.8)$$

$$Q_4^{\Delta s=2} = (\bar{s}_\alpha d_\alpha)(\bar{s}_\beta d_\beta) + (\bar{s}_\alpha \gamma_5 d_\alpha)(\bar{s}_\beta \gamma_5 d_\beta), \quad (3.9)$$

$$Q_5^{\Delta s=2} = (\bar{s}_\alpha \sigma_{\mu\nu} d_\beta)(\bar{s}_\alpha \sigma_{\mu\nu} d_\beta), \quad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \quad (3.10)$$

The parity conserving part (denoted by a superscript “+”) of the operators (3.1)-(3.5) can be written in terms of the operators (3.6)-(3.10)

$$(27, 1) \quad [O_1^{\Delta s=2}]^+ = Q_1^{\Delta s=2} \quad (3.11)$$

$$(6, \bar{6}) \quad \begin{cases} [O_2^{\Delta s=2}]^+ = Q_4^{\Delta s=2} \\ [O_3^{\Delta s=2}]^+ = -\frac{1}{2}(Q_4^{\Delta s=2} - Q_5^{\Delta s=2}) \end{cases} \quad (3.12)$$

$$(8, 8) \quad \begin{cases} [O_4^{\Delta s=2}]^+ = Q_3^{\Delta s=2} \\ [O_5^{\Delta s=2}]^+ = -\frac{1}{2}Q_2^{\Delta s=2} \end{cases} \quad (3.13)$$

It follows from the above considerations that  $Q_1^{\Delta s=2}$  renormalizes multiplicatively,  $Q_2^{\Delta s=2}$  mixes with  $Q_3^{\Delta s=2}$  and  $Q_4^{\Delta s=2}$  mixes with  $Q_5^{\Delta s=2}$ . We denote the renormalization factors computed in this basis by  $Z_{ij}^{\Delta s=2}$ . Moreover, in the  $SU(3)$  flavor limit they are some relations between the renormalization factors of the  $\Delta s = 2$  operators (3.6)-(3.10) and those of the  $\Delta s = 1$  operators (2.1). For example  $O_1^{\Delta s=2}$  and  $Q'_1$  have the same renormalization factor, and the two by two renormalization matrix of  $(Q_7, Q_8)$  is related to the one of  $(Q_2^{\Delta s=2}, Q_3^{\Delta s=2})$  in the following way

$$\begin{aligned} Z_{77}^{\Delta s=1} &= Z_{22}^{\Delta s=2} & Z_{87}^{\Delta s=1} &= -\frac{1}{2}Z_{32}^{\Delta s=2} \\ Z_{87}^{\Delta s=1} &= -2Z_{32}^{\Delta s=2} & Z_{33}^{\Delta s=1} &= Z_{88}^{\Delta s=2} \end{aligned} \quad (3.14)$$

<sup>1</sup>In the literature, we sometimes find the notation VLL for  $O_1$ , SLL for the set  $(O_2, O_3)$  and LR for the set  $(O_4, O_5)$ .

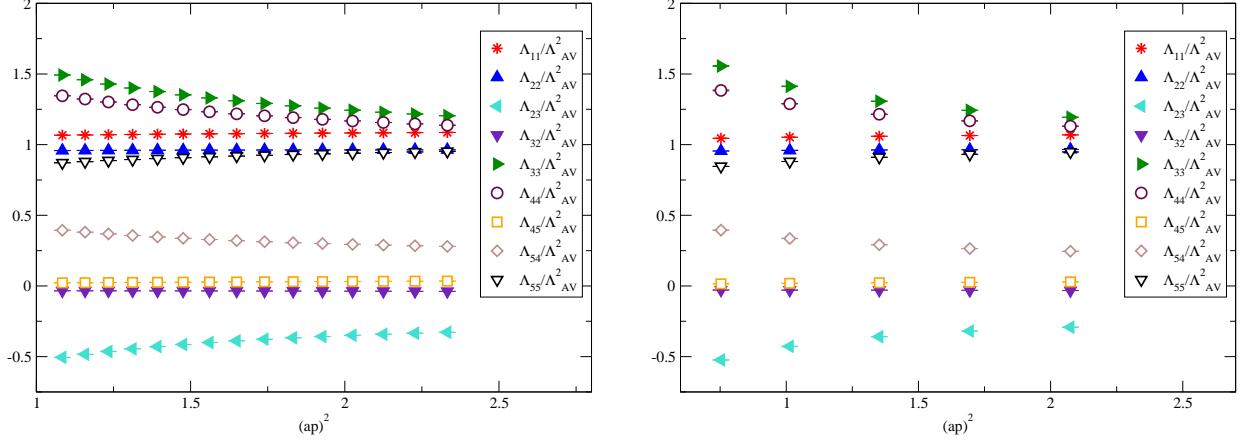
#### 4. Numerical implementation and preliminary results

The numerical setup of this computation is the same as the one presented in details in two recent publications [1, 13]. We use  $n_f = 2 + 1$  flavors of domain wall fermion on a Iwasaki gauge action at two values of the lattice spacing  $a \sim 0.086$  fm and  $a \sim 0.114$  fm, corresponding to the lattice volumes  $32 \times 64 \times 16$  and  $24 \times 64 \times 16$ , respectively. On each ensemble the strange sea quark mass is fixed, while several values of the light sea quark masses have been considered (the corresponding unitary pion mass varies in the range 290 – 420 MeV). In this work we consider only the light valence quark masses which have the same values as their corresponding sea quarks, and perform the chiral extrapolations linearly in the quark mass. This computation was done with 20 configurations, and 100 bootstrap samples. In addition to the standard RI-MOM scheme, we implement also a scheme with non-exceptional (and symmetric) kinematic, which exhibits a better infrared behavior<sup>2</sup> [14, 15]. We also employ momentum sources [16] in order to obtain small statistical errors despite the expensive cost of the quark discretization. Furthermore, we use twisted periodic boundary conditions, which allows us to change smoothly the magnitude of the momentum without changing its direction (and thus control the  $O(4)$  discretization effects) [17]. This setup has been recently used for the computation of  $Z_{B_K}$  [1]. Here we generalize this computation to the operators relevant for neutral kaon mixing beyond the standard model, and to the  $\Delta I = 3/2$  part of  $K \rightarrow \pi\pi$  decay. As described in [18], the  $Z$  matrix is essentially the invert of  $\Lambda_{ij} = P_j \{O_i\}$ , where  $O_i$  is a four-quark operator which belong to the basis given in eqs (3.6)-(3.10).  $P_j$  projects onto the Dirac and color structure of the operator  $O_j$  (and a given flavor structure which depends on the choice of external states).

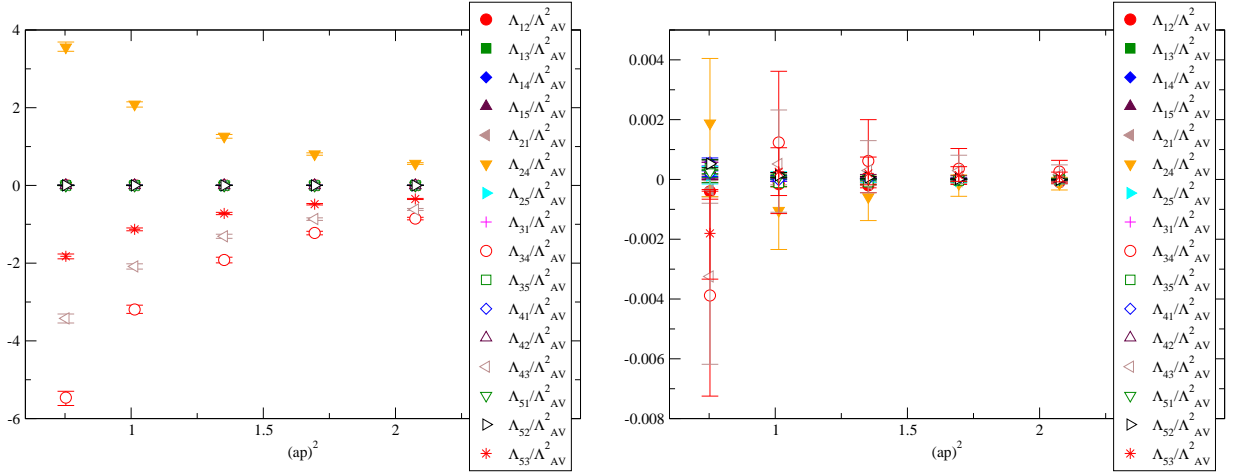
In figures 3 and 4 we show the normalized Green vertex function  $\Lambda_{ij}$  extrapolated to the chiral limit. The range of simulated momenta corresponds to  $1.98 \text{ GeV} \leq p \leq 3.29 \text{ GeV}$  on the finest lattice and to  $1.80 \text{ GeV} \leq p \leq 2.64 \text{ GeV}$  on the coarser one. As expected the use of the momentum sources give us access to a very high statistical precision: at a given momentum the statistical error is below the permille ( $\sim 10^{-4}$  for  $Z_{B_K}$ ). Thanks to the twisted boundary conditions, the  $Z$  factors are smooth functions of the momentum (no scatter coming from the  $O(4)$  discretization effects is visible). Finally the effect of the non-exceptional kinematic is clearly visible in figure 4. The matrix elements shown in these plots should be zero if chiral symmetry was exact. With the Domain-Wall fermions, this is true only in the limit  $L_s \rightarrow \infty$ , so in practice one has to check whether the effects of chiral symmetry breaking can be seen within the numerical precision. For the non-perturbative renormalization, this can be complicated by the presence of some Goldstone poles, which can affect the vertex functions. These poles are suppressed by the use of a non-exceptional kinematic. We confirm here an effect already seen in [4], that the good properties of the Domain Wall action can be obscured by a poor choice of kinematic.

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<sup>2</sup>The non-exceptional scheme implemented here is called SMOM( $\gamma_\mu, \gamma_\mu$ ) in [1]



**Figure 3:** Physical Green vertex functions for the non exceptional kinematic (SMOM scheme), in the chiral limit. On the left panel we show the results for the  $24^3$  lattice ( $a \sim 0.114$  fm) and on the right for the  $32^3$  lattice ( $a \sim 0.086$  fm). The error bars are smaller than the symbols.



**Figure 4:** Chirally disallowed Green vertex functions on the finest lattice for the exceptional (left) and non-exceptional (right) kinematic. We see that for the exceptional kinematic some of these matrix elements are suppressed only at high momenta, while in the non exceptional case they are all zero within the statistical precision.

## 5. Conclusions and outlook

By combining momentum source with twisted boundary conditions and non exceptional kinematic we can obtain the renormalization factors of kaon four-quark operators with a very good handle on the different kinds of errors: the statistical errors are tiny (below the permille), most of

the unwanted infrared effects are suppressed and the usual scatter coming from the  $O(4)$  discretization errors is absent. Even with this precision, when a non-exceptional kinematic is implemented, the non-physical mixing of the four-quark operators is compatible with zero, thanks to the good chiral properties of the Domain Wall action. We are currently extending our computation to a larger physical volume discussed in [9], where the simulated pion mass is significantly smaller (down to 180 MeV). We plan to use the step scaling method introduced in [17] in order to enlarge the Rome-Southampton window. We have also started a computation of the eye diagrams, with the use of stochastic sources.

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